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Every T-Space Is Equivalent to a T-Space of Continuous Functions

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A T-space U of degree k is a $(k+1)$-dimensional vector space over $\mathbb{R}$ (the real line) of real-valued functions defined on a linearly ordered set, satisfying the condition: for every nonzero $u \in U$, $Z(u)$, the number of distinct zeros of u and $S^-(u)$, the number of alternations in sign of $u(t)$ with increasing t, each do not exceed k. It is demonstrated that given a T-space U of degree $k > 0$ on an arbitrary linearly ordered set T, there is a subset T' of the real line and a nonsingular linear map $L: U \to C(T')$, the set of continuous functions on T', such that the following hold: $L(U)$ is a T-space of degree k; for $u \in U$, $Z(u) = Z(L(u))$, $S^-(u) = S^-(L(u))$; and for some order-preserving bijection $\Theta: T \to T'$, $u(t) = 0$ if and only if $L(u)(\Theta(t)) = 0$. It is also shown that a T-space on a subset $T \subset \mathbb{R}$ can be extended to a T-space on the closure of T in $]\inf T, \sup T]$, provided that there are no "interval gaps" in T. Examples show that, in general, a T-space cannot be extended across an "interval gap" in its domain, and cannot be extended to both the infimum and supremum of its domain. Conditions for a T-space to be Markov, and to admit an adjoined function are derived.

1. INTRODUCTION

A T-space U of degree k (see [1]) is a $(k+1)$-dimensional vector space over $\mathbb{R}$ (the real line) of real-valued functions defined on a linearly ordered set T, satisfying the condition: for every nonzero $u \in U$, $Z(u)$, the number of distinct zeros of u and $S^-(u)$, the number of alternations in sign of $u(t)$ with inceasing t, each do not exceed k. Any basis $\{u_i\}_{i=0}^k$ of a T-space is a T-system, as classically defined in terms of the permanence of sign and nonvanishing of the Haar determinant $\det(u_i(t_j))$ for all $t_0 < t_1 < \cdots < t_k$, and conversely, if $\{u_i\}_{i-0}^k$ is a T-system, then its linear span is a T-space. If U is a T-space of degree k and $u \in U$ satisfies $Z(u) = k$, then u has no alternations in sign without an intervening zero (which is indicated by writing $S^0(u) = 0$; see [1, (3.4)]). A sequence of points $t_0 < t_1 < \cdots < t$ is said to

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form a *weak alternation sequence* of length n for a real-valued function u provided $(-1)^{i-j} u(t_i) u(t_j) \geqslant 0$ for all i,j. $S^+(u)$ is defined to be the supremum over all n such that there is weak alternation sequence of length n for u. A $(k+1)$-dimensional vector space U is a T-space if and only if for each $0 \neq u \in U$, $S^+(u) \leqslant k$ ([1, (3.4)].

A T-system $u_0, \ldots, u_k$ is called a *Markov system* provided that for each $i = 0, 1, \ldots, k$, $u_0, \ldots, u_i$ is a T-system. A T-space of degree k admitting of a Markov system is called a *Markov space* of degree k.

Various authors (e.g. [3–5]) have considered T-spaces of functions whose domains are arbitrary linearly ordered sets. However, the domain of the elements of a nontrivial T-space, it will be seen, must be order isomorphic to a subset of $\mathbb{R}$, and hence it may as well be assumed that the domain is a subset of $\mathbb{R}$. Furthermore, while there are a priori no conditions of continuity on the elements of U, nonetheless the inherent T-space structure provides the elements of U with a mutual regularity which is akin to continuity except for multiplication of every element by a single arbitrary positive function. Hence it will be shown that every T-space of degree greater than zero on an arbitrary linearly ordered set is isomorphic to a T-space of continuous functions on a subset of the real line by an isomorphism which preserves relative positions of sign alternations and zeros. Such an isomorphism will be called an *equivalence* in what follows.

Given a T-space U of functions on a subset $T \subseteq \mathbb{R}$, there is a continuous monotonic $m: \mathbb{R} \to \mathbb{R}$ such that $m(T)$ is bounded and $U \circ m^{-1} \equiv \{u \circ m^{-1} \mid u \in U\}$ is a T-space on $m(T)$. Hence, it may be assumed without loss of generality in this paper that every domain $T \subseteq \mathbb{R}$ is bounded.

Let $u_0(t) = \sin t$, $u_1(t) = \cos t$ for $t \in]0, \pi[= T$. Then it is clear that the two dimensional linear space U, spanned by u_0, u_1 is a T-space of degree 1 on T. It is easy to verify that the domain of U can be extended to include $0 = \inf T$ or $\pi = \sup T$. However there is no T-space on $[0, \pi]$ whose restriction to T is U. We will show in what follows that, in general, the domain of a T-space, T, can be extended to include one of its extreme points if they are both not in T. Using this result we will show that any T-space on a set that does not contain its extreme points is Markov. (This last result has been proved independently by Zalik [6].) Prviously, this was only known for sets with the additional property that between every two points is a third ("property D"; see [4]).

Furthermore we show that it is possible to extend a T-space on $T \subseteq \mathbb{R}$ to a point in its domain provided the gap is a closed interval (a provision shown by an example to be necessary). Using this we show that any T-space on a domain with "property D" derives from a T-space on an open interval through the restriction of the elements to a subset of that interval. In particular Theorem 3 of Zielke [5], which states that any Markov space of degree greater than zero on a set with "property D" is extendable to a

Markov space of higher degree, now follows directly from the result of Rutman [3], which states the same thing for Markov spaces on open intervals.

A further consequence is derived from the above results: Given a real-valued function u defined on a subset $T \subseteq \mathbb{R}$, $t \in \mathbb{R}$ has been called an *asymptotic zero of u* if there exists a sequence $(t_n) \subseteq T$ such that $t_n \to t$ and $u(t_n) \to 0$ (see [2]). Let $AZ(u)$ denote the number of distinct asymptotic zeros of u in $]\inf T, \sup T[$ which are not also actual zeros of u. It is shown that, after multiplying each element by a single positive function, a T-space U of degree k on an interval has the property: $0 \neq u \in U$ implies $Z(u) + AZ(u) \leqslant k$.

Throughout, it should be understood that left-biased results can be restated as the corresponding right-biased results, even though this is not explicitly stated. We use 1 to denote the function $u: T \to \mathbb{R}$ satisfying $u(t) = 1$ for all $t \in T$; we define id: $\mathbb{R} \to \mathbb{R}$ by $id(t) = t$ for all $t \in T$. If $u_0, ..., u_k$ is a Markov system on T then u_0 is either strictly positive or strictly negative and $1, (1/u_0) u_1, ..., (1/u_0) u_k$ is also a Markov system on T. If $1, u_1, ..., u_k$ is a Markov system on T $(k > 0)$ then u_1 is strictly monotone and $1, id, u_2 \circ u_1^{-1}, ..., u_k \circ u_1^{-1}$ is also a Markov system on $u_1(T)$.

We recall the following Lemma:

4.1. LEMMA OF [1]. *Suppose* $T \subset \mathbb{R}$, card $T > k$ *and* $x_1, ..., x_k \in \mathbb{R}$ *satisfy* $x_1 < \cdots < x_k < \inf T$. *Then any T-space of degree k on* $\{x_1, ..., x_k\} \cup T$ *is a Markov space of degree k on T.*

1.1. THEOREM. *If an arbitrary linearly ordered set T is the domain of a T-space of degree greater than zero, then T is order-isomorphic to a subset of* $\mathbb{R}$.

Proof. Given a T-space U of degree $k > 0$ on an arbitrary linerly ordered set T, suppose that T has infinite cardinality and find $x \in T$ such that for $T_- \equiv \{t \in T \mid t < x\}$ and $T_+ \equiv \{t \in T \mid t > x\}$, card $T_- > k$ and card $T_+ > k$. The restriction $U|_{T_+}$ of the elements of U to T_+ is a Markov space by the lemma above.

Thus there is a strictly monotone map $\alpha: T_+ \to \mathbb{R}$ with $\inf \alpha(T_+) > 0$. Similarly, there is a strictly monotone map $\beta: T_- \to \mathbb{R}$ with $\sup \beta(T_-) < 0$, whence there is a strictly increasing map $m: T \to \mathbb{R}$.

Note. This cannot be inferred merely from the knowledge that card $T \leqslant$ card $\mathbb{R}$, as there are linearly ordered sets of cardinality equal to that of $\mathbb{R}$ which cannot be mapped in an order-preserving way into $\mathbb{R}$: the first uncountable ordinal, for example.

2. Extending the Domain

The possibility of extending the domain of a T-space U to one of its extreme points is related to the existence of T-subspaces in U.

2.1. Lemma. *A T-space U of degree k contains one of degree $k-1$ if and only if there is a T-space V of degree k on $T \cup \{x\}$, $x > \sup T$, whose restriction to T is U.*

Proof. Suppose V is a T-space of degree k on $T \cup \{x\}$ such that $V|_T = U$. Consider the set U' of elements of V that vanish at x. Clearly for any $0 \neq v \in U'$, $Z(v|_T) \leqslant k-1$, since $Z(v) \leqslant k$. Furthermore $S^-(v|_T) \leqslant k-1$, since $S^+(v) \leqslant k$. Hence $U'|_T$ is a T-space of degree $k-1$ contained in U.

Conversely let $u_0, \ldots, u_{k-1}, u_k$ be a basis for U such that $u_0, \ldots, u_{k-1}$ is a T-system. Let $x > \sup T$, define $v_i(t) = u_i(t)$ for $t \in T$ and let $v_i(x) = \delta_{ik}$ $(i = 0, \ldots, k)$. It is easily verified that the space V spanned by v_i, $i = 0, 1, \ldots, k$ is a T-space on $T \cup \{x\}$.

2.2. Lemma. *Let $T \subseteq \mathbb{R}$ with* $\inf T$, $\sup T \notin T$, *and U any T-space on T. Then there exists a T-space V on $T \cup \{\sup T\}$ of the same degree as U such that V coincides with U on T.*

Proof. Let k be the degree of U and let $u_0, u_1, \ldots, u_k$ be a basis for U. Define $\phi(t) = 1/\sum_{i=0}^{k} |u_i(t)|$, $t \in T$. ϕ is well defined since all u_i cannot simultaneously vanish at any $t \in T$ (this follows from the nonvanishing of the Haar determinant). Let $t_n \to \sup T$ be a sequence in T such that $c_i = \lim \phi(t_n)\, u_i(t_n)$ exists for $i = 0, \ldots, k$. Since $\Sigma |c_i| = 1$, at least one $c_i \neq 0$. Clearly since $\phi(t) > 0$, U is a T-space iff $\{\phi \cdot u \mid u \in U\}$ is a T-space. So we can assume without loss of generality that $\phi(t) = 1$. Furthermore we can assume the u_i's to be such that $c_0 > 0$. We will now prove that the required T-space V is spanned by functions v_i defined for $t \in T$ by $v_i(t) = u_i(t)$ and satisfying $v_i(\sup T) = c_i$ $(i = 0, \ldots, k)$.

First we show that $S^-(v) \leqslant k$ for each $v \in V$. Suppose $S^-(v) > k$ for some $v \in V$. We may then assume that there exist points $s_0 < s_1 < \cdots < s_k$ in T such that $(-1)^{k-j} v(s_j) < 0$ and $v(\sup T) > 0$, since $v|_T \in U$ implies $S^-(v|_T) \leqslant k$. Since $v(\sup T) = \lim v(t_n)$, there exists some $t_n > s_k$ such that $v(t_n)$ has the same sign as $v(\sup T)$. But this cannot be since $S^-(v|_T) \leqslant k$. Therefore for each $v \in V$, $S^-(v) \leqslant k$.

Next we show $Z(v) \leqslant k$ for $0 \neq v \in V$. Suppose this is not so. Then there exist $v \neq 0$ in V and points $s_1 < s_2 \cdots < s_k$ such that $v(s_i) = 0$, $i = 1, \ldots, k$ and $v(\sup T) = 0$, since $Z(v|_T) \leqslant k$. Let $s_0 < s_1$ be an element of T (such an s_0 exists since $\inf T \notin T$). Further from the nonvanishing of the Haar determinant we may conclude that there exists $u \in U$ such that $(-1)^{k-i} u(s_i) > 0$,

$i=0, 1,\ldots, k$. Since $S^-(u) \leqslant k$, $\lim u(t_n) \geqslant 0$. If $\lim u(t_n)=0$, then by choosing $\eta > 0$ sufficiently small, $(-1)^{k-i}(u+\eta u_0)(s_i)>0$ and $\lim(u(t_n)+\eta u_0(t_n))=\eta c_0>0$. Thus, we may assume that $\lim u(t_n)>0$. Furthermore, for some m, $t_m > s_k$ and either $-v(t_m)$ or $v(t_m)$ is positive. Since $Z(v)=Z(-v)$ we might as well assume $v(t_m)<0$. Then there exists a ρ sufficiently large such that $(-1)^{k-i}(u+\rho v|_T)(s_i)>0$, $i=1,\ldots,k$ and $(u+\rho v|_T)(t_m)<0$. Since $\lim u(t_n)>0$ and $\lim v(t_n)=0$, there exists $t_{m'}$ such that $t_m < t_{m'} < \sup T$ and $(u+\rho v|_T)(t_{m'})>0$. Hence the sign of $u+\rho v|_T$ alternates between each of the points $s_1, s_2,\ldots, s_k, t_m, t_{m'}$. Therefore $S^-(u+\rho v|_T)>k$, a contradiction.

2.3. THEOREM. *If a T-space U on* $T \subseteq \mathbb{R}$ *is not a Markov space then either* $\inf T \in T$ or $\sup T \in T$.

Proof. Suppose $\inf T$, $\sup T \notin T$. By (2.2), U can be extended to a T-space on $T \cup \{\sup T\}$. From (2.1) therefore there exists a subspace U' of U that is a T-space of degree $k-1$. We can now apply the same argument to U' and so on, deriving a nested chain of T-spaces in U which implies U is Markov. Hence the theorem follows.

2.4. COROLLARY. *If U is a T-space on* $T \subseteq \mathbb{R}$ *with* $\inf T$, $\sup T \notin T$ *then there is a* $u \in U$ *such that* $u(t)>0$ *for* $t \in T$.

Even if $\inf T$, $\sup T \in T$, it is possible to extend T–spaces to include points belonging to certain kinds of "gaps" in T. This is made precise in the next theorem. On the other hand, there are examples of "gaps" in T in which the given T-space cannot be extended to even one point therein.

There are trivial examples of Markov spaces wherein the domain cannot be extended to internal points when the domain is not closed. For example, $u_0=1$, $u_1(t)=t$ for $t \leqslant 0$ and $u_1(t)=t-1$ for $t>1$ form a Markov system of continuous functions on $\mathbb{R}\backslash]0,1]$ which cannot be extended to any point of $]0,1]$. However, there are also examples of Markov spaces of continuous functions on a closed set which cannot be extended to a continuum of internal points. For example, let $T=[0,\pi/2] \cup [5\pi/2, 3\pi]$ and on T define $u_0(t)=1$, $u_2(t)=\sin t$, $u_1(t)=1+\cos t$ if $t \leqslant \pi/2$ and $u_1(t)=-1+\cos t$ otherwise. It is not hard to see that u_0, u_1, u_2 form a Markov system on T (if U is their linear span, $U \circ u_1^{-1}$ is an isomorphic space generated by $1, t$ and $u_2 \circ u_1^{-1}$, and $u_2 \circ u_1^{-1}$ is strictly increasing for $t \leqslant \pi/2$ and strictly decreasing thereafter). On the other hand, given $x \in]\pi/2, 5\pi/2[$, there are no values for $u_0(x)$, $u_1(x)$, $u_2(x)$ such that the corresponding extensions of the elements of U form even a T-space on $T \cup \{x\}$. The demonstration of this involves a geometric argument applied to the curve $\{(u_0(t), u_1(t), u_2(t)) \mid t \in T\}$, which is laborious and hence omitted.

2.5. LEMMA. *Let X be a T-space of functions defined on a subset $T \subseteq \mathbb{R}$, and suppose $1 \in X$. Then for each $u \in X$ and each s in the closure of T, $\lim_{t\uparrow s} u(t)$ (respectively, $\lim_{t\downarrow s} u(t)$) exists or is infinite if s is an accumulation point of T from the left (respectively, right).*

Proof. If $\underline{\lim}\, u(t) < b < \overline{\lim}\, u(t)$ (as, say, $t \uparrow s$) then $S^+(u - b \cdot 1) = \infty$.

2.6. LEMMA. *Let $1 = u_0, u_1, ..., u_k$ be a Markov system defined on a subset $T \subset \mathbb{R}$. Then for $\inf T < a < b < \sup T$, each of $u_1, ..., u_k$ is bounded on $[a, b] \cap T$.*

Proof. Assume not. By (2.5) and the compactness of $[a, b]$, it may be assumed that for some $s \in \,]\inf T, \sup T[$ and some $n > 0$ $u_0, u_1, ..., u_{n-1}$ are bounded in a neighborhood of s, but (say) $\lim_{t\uparrow s} u_n(t) = \infty$. Let $v \neq 0$ be a linear combination of $u_0, ..., u_n$ with n zeros $t_0 < \cdots < t_{n-1} < s$, normalized so that $\lim_{t\uparrow s} v(t) = \infty$. Let $u \neq 0$ be a linear combination of $u_0, ..., u_{n-1}$ with zeros at $t_0, ..., t_{n-2}$ and such that $u(t_{n-1}) > 0$. Let $t_{n+1} \in T \cap \,]s, +\infty[$ and find $\alpha > 0$ such that $\alpha u(t_{n+1}) > v(t_{n+1})$. Find $t_n \in T \cap \,]t_{n-1}, s[$ such that $\alpha u(t_n) < v(t_n)$. Then $t_0, ..., t_{n+1}$ is a weak alternation sequence for $v - \alpha u$, so $S^+(v - \alpha u) \geqslant n + 1$, which contradicts that $u_0, ..., u_n$ generate a T-space.

Remark. Suppose U is a T-space on $T \subset \mathbb{R}$ and $1 \in U$. Then for each accumulation point s of T there are $a, b \in \mathbb{R}$ with $a < s < b$ such that U restricted to $[a, b] \cap T$ is a Markov space (containing 1). However, it is possible that no Markov system generating that Markov space has 1 as its first element in which case its elements need not be bounded in any neighborhood of s. For example, $u_0(t) = t^{-2}$, $u_1(t) = t^{-1}$, $u_2(t) = 1$ is a Markov system on $\mathbb{R}\backslash\{0\}$, unbounded in every neighborhood of 0.

2.7. THEOREM. *Suppose U is a T-space on $T \subset \mathbb{R}$ and for some $x \in \,]\inf T, \sup T[$, neither $\sup\{t \in T \mid t < x\}$ nor $\inf\{t \in T \mid t > x\}$ are elements of T. Then there is a T-space V on $T \cup \{x\}$ whose restriction to T is U.*

Proof. Suppose the degree of U is k. The hypothesis is vacuous unless there are $a, b \in T$ such that $a < x < b$ and $A \equiv T \cap [a, b]$ is infinite but has at least $2k$ fewer points than T. In this case the restriction $U|_A$ is a Markov space by (4.1) of [1]. Thus there is a strictly positive p on A such that $1, u_1, ..., u_k$ is a Markov system generating $p \cdot U|_A$. Let $y = \sup\{t \in T \mid t < x\}$. By (2.5) and (2.6), for each $u \in U$, $\lim_{t\uparrow y} p(t) \cdot u(t)$ exists. If $x \notin T$, extend each $u \in U$ to $\hat{u}$ on $T \cup \{x\}$ by defining $\hat{u}(x) = \lim_{t\downarrow y} p(t) \cdot u(t)$. Notice that if $\hat{u}(x) \neq 0$ then for all $t \in T$, $t < y$, t sufficiently close to y, the sign of $\hat{u}(x)$ is the same as that of $\hat{u}(t) = u(t)$. Thus $S^-(\hat{u}) \leqslant k$. It remains to show that if $Z(u) = k$ then $\hat{u}(x) \neq 0$ (whence for all $0 \neq u \in U$, $\max\{S^-(\hat{u}), Z(\hat{u})\} \geqslant k$

which proves that the set of extensions $\{\hat{u} \mid u \in U\}$ forms a T-space on $T \cup \{x\}$).

Indeed, suppose both $Z(u) = k$ and $\hat{u}(x) = 0$. Let us first assume that the k zeros of u are $t_0 < t_1 < \cdots < t_r < t_{r+4} < \cdots < t_{k+2}$ with $x \in]t_r, t_{r+4}[\cap T$. From the assumptions of the theorem there are infinitely many points of T in $]t_r, x]$ and in $]x, t_{r+4}[$. Let $s_1, s_2 \in]t_r, x[\cap T$ such that $]t_r, s_1[\cap T$, $]s_1, s_2[\cap T \neq \varnothing$. Since U is a T-space it follows from the nonvanishing of the Haar determinant that there exists a $v \in U$ such that at the $k+1$ points $t_0, t_1, \ldots, t_r, s_1, s_2, t_{r+4}, \ldots, t_{k+1}$, v assumes the values:

$$(-1)^{j+r} v(t_j) = -1, \qquad j = 0, 1, 2, \ldots, r, r+4, \ldots, k+1,$$
$$v(s_1) = v(s_2) = 0.$$

Then $v(t) < 0$ on $]s_2, t_{r+4}[\cap T$: if for some $s \in]s_2, t_{r+4}[\cap T$, $v(s) \geqslant 0$, $t_0, t_1, \ldots, t_r, s_1, s_2, s, t_{r+4}, \ldots, t_{k+1}$ forms a weak alternation sequence of length $k+1$ for v. Thus $\hat{v}(x) \leqslant 0$. Since $1 \in p \cdot U|_A$, there exists an $\omega \in U$ such that $\omega(t) < 0$ for all $t \in A$ and $\hat{\omega}(x) < 0$. There exists an $\eta > 0$ such that

$$(-1)^{j+r}(v(t_j) + \eta\omega(t_j)) < 0, \qquad j = 0, 1, 2, \ldots, r, r+4, \ldots, k+1.$$

Let $f = v + \eta\omega$ and pick points $t_{r+1} \in]t_r, x[\cap T$ and $t_{r+3} \in]x, t_{r+4}[\cap T$. Then $t_0, \ldots, t_r, t_{r+1}, t_{r+3}, t_{r+4}, \ldots, t_{k+2}$ is a weak alternation sequence of length $k+1$ for u unless $u(t_{r+1})\, u(t_{r+3}) > 0$, which thus must be the case. We may consequently assume that $u(t) > 0$ for all $t \in]t_r, t_{r+4}[$. Hence there exists $\rho > 0$ such that

$$(-1)^{j+r}(f(t_j) + \rho u(t_j)) < 0, \qquad j = 0, 1, \ldots, r+1, r+3, \ldots, k+1$$

(since $u(t_j) = 0$ if $j \neq r+1, r+3$). Since $\hat{f}(x) < 0$ while $\hat{u}(x) = 0$ there exists $t_{r+2} \in]t_{r+1}, t_{r+3}] \cap T$ such that $f(t_{r+2}) + \rho u(t_{r+2}) < 0$. Hence

$$(-1)^{j+r}(f(t_j) + \rho u(t_j)) < 0, \qquad j = 0, 1, \ldots, k+1,$$

i.e., $S^-(f + \rho u) \geqslant k+1$, which cannot be since $f + \rho u \in U$.

Finally, assume that all the zeros of u lie to one side of x. If $k = 0$ then $u|_A = \alpha/p$ for some scalar $\alpha \neq 0$ and $\hat{u}(x) = \alpha$. Hence, assume $k > 0$ and $t_0 < t_1 < \cdots < t_{k-1}$ are the zeros of u, with (say) $t_{k-1} < x$. Let $s \in]t_{k-1}, x[\cap T$. As above, there exists a $v \in U$ such that $(-1)^{k-i} v(t_i) > 0$, $i = 0, \ldots, k-1$ and $v(s) > 0$. Since $S^+(v) \leqslant k$, $v(t) > 0$ for all $t \in [s, \sup T[\cap T$ and thus $\hat{v}(x) \geqslant 0$. As in the proof of (2.2), we may assume that $\hat{v}(x) > 0$. Let $t_{k+1} \in]x, \sup T[\cap T$. Since $S^+(u) \leqslant k$, $u(t)\, u(t_{k+1}) > 0$ for all $t \in]t_{k-1}, t_{k+1}[\cap T$. Replacing u by $-u$ if necessary, we may assume that $u(t_{k+1}) > 0$. Let $\delta > 0$ be chosen such that $\delta v(t_{k+1}) < u(t_{k+1})$. Then $\delta\hat{v}(x) > 0$ while $\hat{u}(x) = 0$ and thus there exists a point $t_k \in \,]s, x[\, \cap T$ such that $\delta v(t_k) > u(t_k)$. Thus $(-1)^{k-i}(\delta v - u)(t_i) > 0$ for $i = 0, \ldots, k+1$ so $S^-(\delta v - u) > k$, a contradiction.

Remark. The techniques used to prove (2.2) and (2.7) are interchangeable, affording two distinct proofs of each result. (However, the differences beween the two results in any case require that two distinct proofs be given.)

3. Continuity

Clearly, individual elements of a T-space can be terribly discontinuous: for any T-space U of functions defined on T and any real-valued function ϕ satisfying $\phi(t) > 0$ for all $t \in T$, $\phi \cdot U \equiv \{\phi(t) \cdot u(t) \mid u \in U\}$ is also a T-space. If $T \subset \mathbb{R}$ and the elements of U are continuous while ϕ is discontinuous, the elements of the T-space $\phi \cdot U$ will inherit the discontinuities of ϕ.

On the other hand, if U is a T-space of functions on a subset $T \subseteq \mathbb{R}$, continuous except for possibly a jump discontinuity at a common point x, then for

$$\begin{aligned} \Theta(t) &= t - 1 && \text{if} \quad t < x \\ &= x && \text{if} \quad t = x \\ &= t + 1 && \text{if} \quad t > x \end{aligned}$$

$U \circ \Theta^{-1} \equiv \{u \circ \Theta^{-1} \mid u \in U\}$ is a T-space on $\Theta(T)$ whose elements are continuous everywhere (the discontinuity having been isolated).

It is shown in this section that any T-space can be derived as above from a T-space of functions continuous on the complement of a countable set, and in fact whose only discontinuities are jumps. Specifically, this is stated as follows.

3.1. Theorem. *Let U be a T-space of degree greater than zero, of real-valued fuctions defined on a linearly ordered set T. Then there exists a strictly monotone real-valued function Θ on T and a strictly positive real-valued function ϕ on $\mathbb{R}$ such that the elements of the T-space $\phi \cdot U \circ \Theta^{-1} \equiv \{\phi(t) \cdot u(\Theta^{-1}(t)) \mid u \in U,\ t \in \Theta(T)\}$ are continuous real-valued functions on $\Theta(T) \subseteq \mathbb{R}$.*

The two T-spaces U and $\phi \cdot U \circ \Theta^{-1}$ are of the same degree and the association $u \to \phi \cdot U \circ \Theta^{-1}$ is an isomorphism wherein the elements of each associated pair have the same alternation properties. In particular, $Z(u) = Z(\phi \cdot u \circ \Theta^{-1})$, $S^-(u) = S^-(\phi \cdot u \circ \Theta^{-1})$ and $u(t) = 0$ if and only if $\phi(t) \cdot u \circ \Theta^{-1}(\Theta(t)) = 0$.

The proof of (3.1) depends upon an additional lemma.

3.2. LEMMA. *Let* $1, id, u_2,\ldots, u_k$ *be a Markov system defined on a subset of* $\mathbb{R}$. *Then each of* $u_2,\ldots, u_k$ *is continuous (except perhaps at one or both extreme points of the domain, if they exist).*

Proof. By induction on k. If $k=1$ then the lemma is vacuous. Now suppose $u_0(t)=1$, $u_1(t)=t$, $u_2(t),\ldots, u_k(t)$ is a T-system on a set $T\subseteq\mathbb{R}$ with $u_0,\ldots, u_{k-1}$ continuous. Let $s\in T$ be a non-extreme point of T and assume u_k is discontinuous at s. Suppose s is an accumulation point of T from the right. Then by (2.5), $\lim_{t\downarrow s} u_k(t)$ exists or is infinite. Suppose $u_k(s)<\lim_{t\downarrow s} u_k(t)$. Let $s=t_1<t_3<t_4<\cdots<t_{k+1}$ be points in T and find $a_0,\ldots, a_{k-1}$ such that $u=\sum_{i=0}^{k-1} a_i u_i$ satisfies $u(t_i)=u_k(t_i)$ for $i=1, 3, 4,\ldots, k+1$ (see [1, (1.2)]). By the continuity of u, there exists $t_2\in\,]t_1, t_3[\,\cap T$ such that $u(t_2)<u_k(t_2)$. If for some $t_0\in T$ with $t_0<t_1$, $u(t_0)\leqslant u_k(t_0)$, then $t_0<t_1<t_2<\cdots<t_{k+1}$ is weak alternation sequence for $u-u_k$ of length $k+1$, which is impossible. Thus for all $t<t_1$, $u(t)>u_k(t)$. Let $t_0\in T$ with $t_0<{}_1$ be chosen and find $b_0,\ldots, b_{k-1}$ such that $v=\sum_{i=0}^{k-1} b_i u_i$ satisfies $v(t_i)=u_k(t_i)$ for $i=0, 1, 4,\ldots, k+1$. Note that $v(t_0)=u_k(t_0)<u(t_0)$. If $v(t_3)\leqslant u(t_3)$ then $t_0<t_1<t_3<t_4<\cdots<t_{k+1}$ is a weak alternation sequence for $u-v$ of length k, which is impossible. Hence $v(t_3)>u(t_3)=u_k(t_3)$. However, by the continuity of v, there exists $t_*\in\,]t_1, t_3[\,\cap T$ such that $v(t_*)<u_k(t_*)$. Thus $t_0<t_1<t_*<t_3<\cdots<t_{k+1}$ is a weak alternation sequence for $v-u_k$ of length $k+1$, also impossible. It follows that u_k must be right continuous at s. Similarly it follows that u_k must be left continuous at s.

Proof of (3.1). By (1.1) it may be assumed that $T\subset\mathbb{R}$. If T is finite, there is nothing left to prove. Hence, assume T is infinite. Suppose the degree of U is k. By (2.3) and [1, (4.1)] there are $a, b\in\mathbb{R}$ such that for $A=\,]a, b[\,\cap T$, $U|_A$ is a Markov space of degree k and $T\backslash A$ contains at most $2k$ elements: if there are at least k isolated points at one end of T, (4.1) of [1] applies; otherwise, after removing fewer than k points each end of T, both the infimum and supremum of what is left are accumulation points and upon their removal (2.3) applies.

Suppose $u_0,\ldots, u_k$ is a Markov system for $U|_A$. The Markov space $V\equiv(1/u_0)\cdot U|_A$ on A is generated by a Markov system of the form $1, v_1,\ldots, v_k$ and the Markov space $W\equiv V\circ u_1^{-1}$ on $u_1(A)$ is generated by a Markov system of the form $1, id, w_2,\ldots, w_k$. Each element of such a Markov space W must be continuous by (3.2). The map $u_1: A\to\mathbb{R}$ admits a strictly monotonic extension $\Theta: T\to\mathbb{R}$. Thus, for any strictly positive extension $\phi\cdot U\circ\Theta^{-1}$ is a T-space of continuous functions on $\Theta(T)$ with the desired properties.

Remark. The effect of the map Θ is simply to "scale" the argument $t\in T$ and to create gaps in the domain; it has no effect on the alternation properties of the elements of U. On the other hand, these gaps are essential to the continuity argument. For example, let $\Theta(t) = t$ if $t \leqslant 0$ and let $\Theta(t) = t + 1$ if $t > 0$. While the linear span U of $1, \Theta$ is a Markov space on $\mathbb{R}$, Θ has a discontinuity. However, while $1, t \in U \circ \Theta^{-1}$ (and thus the elements of $U \circ \Theta^{-1}$ are continuous) the gap $]0, 1]$ appears in the domain. Rectifying the discontinuity in Θ via multiplication by a Θ would destroy the continuity of 1 and there can be no reconciliation of the gap and continuity. Thus, for example, it is false that every T-space (or even Markov space) on an open interval is equivalent to a T-space of continuous functions *on an open interval.*

We will say that a set $S \subseteq \mathbb{R}$ is *pre-closed* if for each $x \in]\inf S, \sup S[$, $\sup\{t \in S \mid t < x\} = \inf\{t \in S \mid t > x\}$ except perhaps when both are elements of S. We will say S is *relatively closed* if $S \backslash \{\inf S, \sup S\}$ is relatively closed in $]\inf S, \sup S[$. A relatively closed set is pre-closed.

3.3. COROLLARY. *Given a T-space U on a linearly ordered set T, there exists a T-space V on a relatively closed subset $S \subset \mathbb{R}$ such that for some strictly increasing $\Theta: T \to S$ the restriction $V|_{\Theta(T)}$ is equivalent to U, i.e., $V|_{\Theta(T)} = U \circ \Theta^{-1}$. In particular, if T has "property D" then S will be an open interval.*

Proof. By (1.1), U is isomorphic to a T-space on a subset $A \subset \mathbb{R}$ and, by applying a strictly increasing transformation to A, it may be assumed that A is pre-closed. Applications of (2.7) show that the equivalent T-space on A may be extended to a T-space V on the closure of A in $]\inf A, \sup A[$. A pre-closed subset of $\mathbb{R}$ with "property D" is a dense subset of an open interval.

3.4. COROLLARY (Zielke). *Let U be a T-space of degree $k > 0$ on a set T with "property D." Then U is a Markov space and there exists an infinite chain of Markov spaces $U \subset U_{k+1} \subset U_{k+2} \subset \cdots$ on T.*

Proof. By (2.3), U is Markov. By (3.5), there is a real open interval S, a strictly increasing $\Theta: T \to S$, and a Markov space V on S such that $V|_{\Theta(T)} = U \circ \Theta^{-1}$. By Theorem 3 of [3], there is an infinite chain of Markov spaces $V \subset V_{k+1} \subset V_{k+2} \subset \cdots$ on S. Setting $U_i = V_i \circ \Theta$ completes the proof.

4. ASYMPTOTIC ZEROS

For the definitions of "asymptotic zero" and $AZ(u)$, the number of asymptotic zeros of a function u, see Section 1. Let U be a T-space of functions defined on a real interval. If $1 \in U$ and ϕ is strictly positive then $\phi \in \phi \cdot U \equiv \{\phi \cdot u \mid u \in U\}$ and $AZ(\phi)$ can be arbitrarily large. On the other hand, if the elements of U are continuous, and $u \in U$ has a zero at x, define $\Theta(t) = t - 1$ when $t < x$, $\Theta(x) = x$ and $\Theta(t) = t + 1$ when $t > x$. Then

$U \circ \Theta^{-1}$ is an equivalent T-space of continuous functions and the image $u \circ \Theta^{-1}$ of u in $U \circ \Theta^{-1}$ not only has a zero at x but has additional asymptotic zeros at $x-1$ and $x+1$. Thus it is possible that $Z(u)+AZ(u)=3k$ even when u is a continuous nonzero element of a T-space of degree k. However, the next result shows that typically these irregularities can be avoided.

4.1. COROLLARY. *Given a T-space U of degree k on a set $T \subseteq \mathbb{R}$ having no one-sided accumulation points in* $]\inf T, \sup T[\cap \mathbb{R}$, *there is a strictly positive function p on T such that for $0 \neq u \in p \cdot U$, $Z(u)+AZ(u) \leqslant k$.*

Proof. As in the proof of (3.1), there are $a, b \in \mathbb{R}$ such that for $A =]a, b[\cap T$, $U|_A$ is a Markov space, and $T \backslash A$ contains at most $2k$ elements. Thus, for some strictly positive p defined on A, $p \cdot U|_A$ admits of a Markov system basis of the form $1 = u_0, \ldots, u_k$. Extend p to be 1 on $T \backslash A$.

Let $x \in]\inf T, \sup T[$ be an asymptotic zero of $0 \neq u \in p \cdot U$. Then the restriction of $p \cdot U$ to $T \backslash \{x\}$ is a T-space which, as in the proof of (2.7), may be extended to a T-space on T in such a way that the newly extended u satisfies $u(x)=0$. In this way asymptotic zeros can be exchanged for actual zeros and the result follows.

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